Benchmark: sensitivity analysis of a four-wheeled buggy vehicle. Step descent maneuver.

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Abstract

The present document describes a four wheeled buggy vehicle with articulated suspensions and the simulation chosen as a benchmark problem for sensitivity analysis of multibody systems.

1 Description of the buggy vehicle

With the purpose of assessing the performance of sensitivity analysis formulations in complex mechanisms, the case study chosen is a buggy vehicle with 14 degrees of freedom shown in Fig.1.

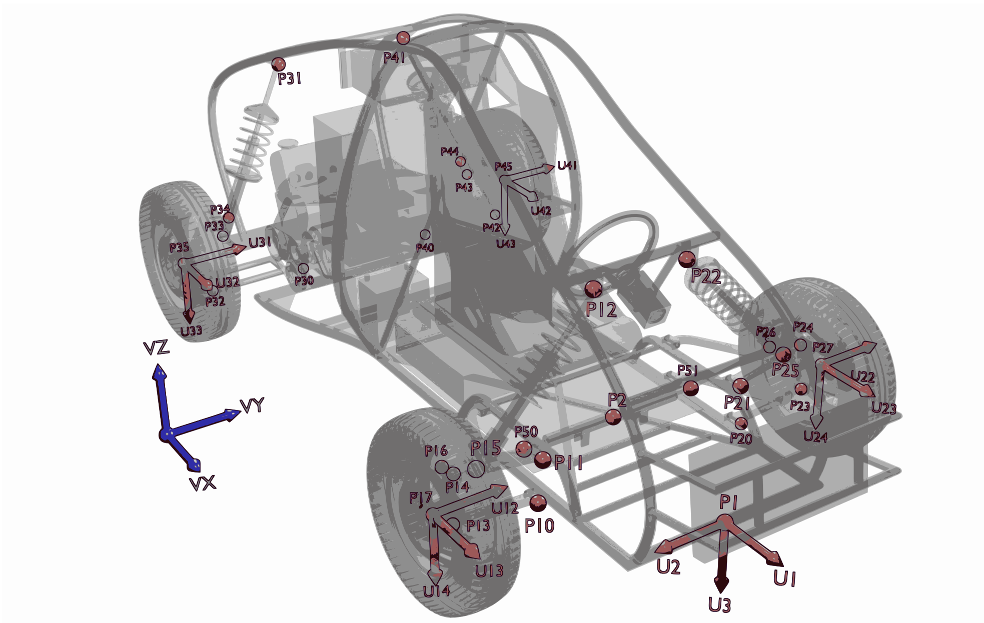

Figure 1: Buggy vehicle.

Due to the large amount of bodies and kinematic relations between them, all the properties of the model are given in table form[1].

- Table 7 gathers the mass properties of the rigid bodies of the model.
- Table 1 and Table 2 list the points and vectors used to describe the model. A point or vector with the same ID that appears in more than one body means that this point or vector is shared between those bodies.
- Some additional auxiliary variables defined in the model are listed in Table 3.
- Table 4 gathers the restrictions needed to define the additional variables and to impose some additional kinematic relationships.

1.1 Kinematic description of the buggy vehicle

In the following tables, the kinematic relations between bodies are presented in terms of shared points and vectors between bodies (points or vectors with identical ID in Tables 1 and 2) and in terms of kinematic constraints (Tables 3 and 4), some of them referred to the evaluation of additional variables like angles and distances.

Body	**Point ID**	**Local Coordinates**			**Global coordinates**		
Ground	81	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Chassis	1	0.00000	0.00000	0.00000	0.00000	0.00000	0.32000
	2	-0.56500	0.00000	-0.07345	-0.56500	0.00000	0.39345
	10	-0.40000	0.39500	0.00000	-0.40000	-0.39500	0.32000
	11	-0.40000	0.38000	-0.14500	-0.40000	-0.38000	0.46500
	12	-0.40815	0.17500	-0.63000	-0.40815	-0.17500	0.95000
	20	-0.40000	-0.39500	0.00000	-0.40000	0.39500	0.32000
	21	-0.40000	-0.38000	-0.14500	-0.40000	0.38000	0.46500
	22	-0.40815	-0.17500	-0.63000	-0.40815	0.17500	0.95000
	30	-2.20500	0.27000	0.00000	-2.20500	-0.27000	0.32000
	31	-2.29000	0.26000	-0.80000	-2.29000	-0.26000	1.12000
	40	-2.20500	-0.27000	0.00000	-2.20500	0.27000	0.32000
	41	-2.29000	-0.26000	-0.80000	-2.29000	0.26000	1.12000
Inferior frontal right wishbone	10	0.02500	0.28000	0.00000	-0.40000	-0.39500	0.32000
	13	0.00000	0.00000	0.00000	-0.42500	-0.65867	0.22577
Superior frontal right wishbone	11	0.02500	0.28000	0.00000	-0.40000	-0.38000	0.46500
	14	0.00000	0.00000	0.00000	-0.42500	-0.65136	0.39598
	15	0.01250	0.07000	0.00000	-0.41250	-0.58352	0.41323
Inferior frontal left wishbone	20	0.02500	0.28000	0.00000	-0.40000	0.39500	0.32000
	23	0.00000	0.00000	0.00000	-0.42500	0.65867	0.22577
Superior frontal left wishbone	21	0.02500	0.28000	0.00000	-0.40000	0.38000	0.46500
	24	0.00000	0.00000	0.00000	-0.42500	0.65136	0.39598
	25	0.01250	0.07000	0.00000	-0.41250	0.58352	0.41323

[1] The model description provided in this document corresponds to the modelization in mixed coordinates (natural plus relative) of the general purpose multibody library MBSLIM [1].

Frontal right upright	13	-0.01500	0.07940	-0.07000	-0.42500	-0.65867	0.22577
	14	-0.01500	0.09054	0.10000	-0.42500	-0.65136	0.39598
	16	-0.13000	0.11000	0.05000	-0.54163	-0.64173	0.34690
	17	0.00000	0.00000	0.00000	-0.40215	-0.73784	0.29389
Frontal left upright	23	-0.01500	-0.07940	-0.07000	-0.42500	0.65867	0.22577
	24	-0.01500	-0.09054	0.10000	-0.42500	0.65136	0.39598
	26	-0.13000	-0.11000	0.05000	-0.54163	0.64173	0.34690
	27	0.00000	0.00000	0.00000	-0.40215	-0.73784	0.29389
Steering rack	50	0.32600	0.00000	0.00000	-0.56500	-0.32600	0.39345
	51	-0.32600	0.00000	0.00000	-0.56500	0.32600	0.39345
Right steering rod	50	-0.16000	0.00000	0.00000	-0.56500	-0.32600	0.39345
	16	0.16000	0.00000	0.00000	-0.54163	-0.64173	0.34690
Left steering rod	51	-0.16000	0.00000	0.00000	-0.56500	0.32600	0.07345
	26	0.16000	0.00000	0.00000	-0.54163	0.64173	0.34690
Frontal right wheel	17	0.03600	0.00000	0.00000	-0.40215	-0.73784	0.29389
Frontal left wheel	27	0.03600	0.00000	0.00000	-0.40215	-0.73784	0.29389
Rear right wishbone	30	0.14500	0.00000	0.17750	-2.20500	-0.27000	0.32000
	32	0.14500	0.00000	-0.17750	-2.20500	-0.60460	0.20140
Rear left wishbone	40	0.14500	0.00000	0.17750	-2.20500	0.27000	0.32000
	42	0.14500	0.00000	-0.17750	-2.20500	0.60460	0.20140
Rear right upright	32	-0.08500	-0.11250	-0.08500	-2.20500	-0.60460	0.20140
	33	-0.17127	0.07117	0.00000	-2.29000	-0.53088	0.39045
	34	-0.20000	0.13500	0.00000	-2.29000	-0.50651	0.45607
	35	0.00000	0.00000	0.00000	-2.29000	-0.69698	0.30792
Rear left upright	42	-0.08500	-0.11250	0.08500	-2.20500	0.60460	0.20140
	43	-0.17127	0.07117	0.00000	-2.29000	0.53088	0.39045
	44	-0.20000	0.13500	0.00000	-2.29000	0.50651	0.45607
	45	0.00000	0.00000	0.00000	-2.29000	0.69698	0.30792
Rear right wheel	35	0.03600	0.00000	0.00000	-2.29000	-0.69698	0.30792
Rear left wheel	45	0.03600	0.00000	0.00000	-2.29000	0.69698	0.30792

Table 1: Buggy model: points.

Body	**Vector ID**	**Local Coordinates**			**Global coordinates**		
Ground	83	0.000	0.000	1.000	0.000	0.000	1.000
Chassis	1	1.000	0.000	0.000	1.000	0.000	0.000
	2	0.000	1.000	0.000	0.000	-1.000	0.000
	3	0.000	0.000	1.000	0.000	0.000	-1.000
Inferior frontal right wishbone	1	1.000	0.000	0.000	1.000	0.000	0.000
	10	0.000	0.000	1.000	0.000	-0.337	0.942
Superior frontal right wishbone	1	1.000	0.000	0.000	1.000	0.000	0.000
	11	0.000	0.000	1.000	0.000	-0.247	0.969
Inferior frontal left wishbone	1	1.000	0.000	0.000	1.000	0.000	0.000
	20	0.000	0.000	1.000	0.000	-0.337	-0.942
Superior frontal left wishbone	1	1.000	0.000	0.000	1.000	0.000	0.000

	21	0.000	0.000	1.000	0.000	-0.247	-0.969
Frontal right upright	12	0.000	1.000	0.000	-0.094	0.995	0.023
Frontal left upright	22	0.000	1.000	0.000	0.094	0.995	-0.023
Steering rack	1	0.000	1.000	0.000	1.000	0.000	0.000
	2	1.000	0.000	0.000	0.000	-1.000	0.000
	3	0.000	0.000	-1.000	0.000	0.000	-1.000
Right steering rod	18	0.000	1.000	0.000	-0.443	-0.163	0.882
	19	0.000	0.000	1.000	-0.894	0.000	-0.449
Left steering rod	28	0.000	1.000	0.000	0.443	-0.163	-0.882
	29	0.000	0.000	1.000	-0.894	0.000	-0.449
Frontal right wheel	12	1.000	0.000	0.000	-0.094	0.995	0.023
	13	0.000	1.000	0.000	-0.996	-0.094	0.004
	14	0.000	0.000	1.000	0.006	-0.022	1.000
Frontal left wheel	22	-1.000	0.000	0.000	0.094	0.995	-0.023
	23	0.000	-1.000	0.000	-0.996	0.094	0.004
	24	0.000	0.000	1.000	0.006	0.022	1.000
Rear right wishbone	1	1.000	0.000	0.000	1.000	0.000	0.000
	30	0.000	1.000	0.000	0.000	0.334	-0.943
Rear left wishbone	1	1.000	0.000	0.000	1.000	0.000	0.000
	40	0.000	1.000	0.000	0.000	0.334	0.943
Rear right upright	1	0.000	0.000	-1.000	1.000	0.000	0.000
	31	-1.000	0.000	0.000	0.000	0.998	0.067
Rear left upright	1	0.000	0.000	1.000	1.000	0.000	0.000
	41	1.000	0.000	0.000	0.000	0.998	-0.067
Rear right wheel	31	1.000	0.000	0.000	0.000	0.998	0.067
	32	0.000	1.000	0.000	1.000	0.000	0.000
	33	0.000	0.000	1.000	0.000	0.067	-0.998
Rear left wheel	41	-1.000	0.000	0.000	0.000	0.998	-0.067
	42	0.000	-1.000	0.000	1.000	0.000	0.000
	43	0.000	0.000	1.000	0.000	-0.067	-0.998

Table 2: Buggy model: vectors.

Description	**Names**	**Initial value**
Front right wheel spring lenght	s_{10}	0.67456
Front left wheel spring lenght	s_{20}	0.67456
Distance between point 50 and point 2	s_{50}	0.32600
Angle of front right wheel	a_{10}	0.00000
Angle of front left wheel	a_{20}	0.00000
Angle of rear right wheel	a_{30}	0.00000
Angle of rear left wheel	a_{40}	0.00000
Rear right wheel spring lenght	s_{30}	0.70822
Rear right wheel spring lenght	s_{40}	0.70822

Table 3: Buggy model: additional variables.

Description	Equation	N
Definition of variable s_{10}: distance between points $\mathbf{p}_{12}$ and $\mathbf{p}_{15}$	$\|\|(\mathbf{p}_{12} - \mathbf{p}_{15})\|\| = s_{10}$	1
Definition of variable s_{20}: distance between points $\mathbf{p}_{22}$ and $\mathbf{p}_{25}$	$\|\|(\mathbf{p}_{22} - \mathbf{p}_{25})\|\| = s_{20}$	2
Prismatic joint: null cross product defined with two points and one vector, and two null dot products between vectors	$\mathbf{v_2} \wedge (\mathbf{p_{50}} - \mathbf{p_2}) = \mathbf{0}$ $\mathbf{v}_2 \cdot \mathbf{v}_{19} = 0$ $\mathbf{v}_2 \cdot \mathbf{v}_{29} = 0$	3 4 5
Definition of variable s_{50}: distance between points $\mathbf{p}_{50}$ and $\mathbf{p}_2$	$\|\|(\mathbf{p}_{50} - \mathbf{p}_2)\|\| = s_{50}$	6
Kinematic guidance of variable s_{50}	$s_{50} = f(t)$	7
Definition of variable a_{10}: angle between vectors $\mathbf{v}_{14}$ and $(\mathbf{p}_{14} - \mathbf{p}_{13})$, measured around $\mathbf{v}_{12}$	${\mathbf{v}_{14}}^T(\mathbf{p}_{14} - \mathbf{p}_{13}) - p_w - p_n cos(a_{10})$ ${\mathbf{v}_{12}}^T((\mathbf{p}_{14} - \mathbf{p}_{13}) \wedge \mathbf{v}_{14}) - p_n \sin(a_{10})$	8 9
Definition of variable a_{20}: angle between vectors $\mathbf{v}_{24}$ and $(\mathbf{p}_{24} - \mathbf{p}_{23})$, measured around $\mathbf{v}_{22}$	${\mathbf{v}_{24}}^T(\mathbf{p}_{24} - \mathbf{p}_{23}) - p_w - p_n cos(a_{20})$ ${\mathbf{v}_{22}}^T((\mathbf{p}_{24} - \mathbf{p}_{23}) \wedge \mathbf{v}_{24}) - p_n \sin(a_{20})$	10 11
Definition of variable a_{30}: angle between vectors $\mathbf{v}_1$ and $\mathbf{v}_{32}$, measured around $\mathbf{v}_{31}$	${\mathbf{v}_1}^T\mathbf{v}_{32} - p_w - p_n cos(a_{30})$ ${\mathbf{v}_{31}}^T(\mathbf{v}_1 \wedge \mathbf{v}_{32}) - p_n \sin(a_{30})$	12 13
Definition of variable a_{40}: angle between vectors $\mathbf{v}_1$ and $\mathbf{v}_{42}$, measured around $\mathbf{v}_{41}$	${\mathbf{v}_1}^T\mathbf{v}_{42} - p_w - p_n cos(a_{40})$ ${\mathbf{v}_{41}}^T(\mathbf{v}_1 \wedge \mathbf{v}_{42}) - p_n \sin(a_{40})$	14 15
Definition of variable s_{30}: distance between points $\mathbf{p}_{31}$ and $\mathbf{p}_{34}$	$\|\|(\mathbf{p}_{31} - \mathbf{p}_{34})\|\| = s_{30}$	16
Definition of variable s_{40}: distance between points $\mathbf{p}_{41}$ and $\mathbf{p}_{44}$	$\|\|(\mathbf{p}_{41} - \mathbf{p}_{44})\|\| = s_{40}$	17
Cross product defined with three points = 0	$(\mathbf{p}_{34} - \mathbf{p}_{33}) \wedge (\mathbf{p}_{31} - \mathbf{p}_{33}) = \mathbf{0}$	18
Cross product defined with three points = 0	$(\mathbf{p}_{44} - \mathbf{p}_{43}) \wedge (\mathbf{p}_{41} - \mathbf{p}_{43}) = \mathbf{0}$	19

Table 4: Buggy model: constraints.

1.2 External forces

The mechanism is subjected to the action of a vertical gravity, $\mathbf{g} = \begin{bmatrix} 0 & 0 & -9.81 \end{bmatrix}^{\mathrm{T}} \mathrm{m\,s^{-2}}$, four forces coming from the suspension spring-dampers, and four contact forces coming from the tire-ground interaction.

1.2.1 Spring-damper suspension forces

The suspension forces are modeled as linear spring-damper forces over the defined distances $(s_{10}, s_{20}, s_{30}, s_{40})$ described in Tables 3 and 4. Their parameters are gathered in Table 5.

N	Variable	Stiffness $(\mathrm{N\,m^{-1}})$	Damping $(\mathrm{N\,s\,m^{-1}})$	l_0 (m)
1	s_{10}	16000	10000	0.71
2	s_{20}	16000	10000	0.71
3	s_{30}	10595	6000	0.741
4	s_{40}	10595	6000	0.741

Table 5: Buggy model: suspension forces (linear spring-damper).

1.2.2 Contact-frictional tire forces

A simple tire model [2], which is composed of normal, longitudinal, and lateral models, is considered in this vehicle model.

For the normal part of the tire forces a linear force model is used:

$$F = -k_m \delta - c_m v_p \tag{1}$$

being k_m and c_m the stiffness and damping coefficients, δ the indentation of the wheel into the ground, and v_p the projected velocity. The projected velocity is computed as:

$$v_p = \mathbf{v} \cdot \mathbf{n} \tag{2}$$

being $\mathbf{v}$ the velocity of the center of the wheel and $\mathbf{n}$ the normal to the ground surface.

The indentation is calculated through a circumference-plane contact model, with each wheel approximated by a circumference of radius 302.53 mm (see Table 6) with its center coincident with points 17, 27, 37 and 47 for the front-right, front-left, rear-right and rear-left wheels, respectively (see Table 1). Normal vectors of each circumference plane are 12, 22, 31 and 41. The width of the wheel is, thus, dismissed in the evaluation of the contact.

The longitudinal and lateral frictional forces are evaluated with linearized models with saturation described as follows:

$$\mathbf{F}_t = F_x \mathbf{b} + F_y (\mathbf{n} \times \mathbf{b}) \tag{3a}$$

$$F_x = \begin{cases} \dfrac{\mu_x |\mathbf{F}_n|}{\kappa_c} \kappa; \; \kappa \leq \kappa_c \\ \mu_x |\mathbf{F}_n|; \; \kappa > \kappa_c \end{cases} \tag{3b}$$

$$F_y = \begin{cases} \dfrac{\mu_y |\mathbf{F}_n|}{\alpha_c} \alpha; \; \alpha \leq \alpha_c \\ \mu_y |\mathbf{F}_n|; \; \alpha > \alpha_c \end{cases} \tag{3c}$$

where $\mathbf{u}$ is the unit vector along the wheel rotation axis, $\mathbf{b} = (\mathbf{u} \times \mathbf{n}) / \|\mathbf{u} \times \mathbf{n}\|$ is the longitudinal vector, κ is the longitudinal slip at the tire-ground contact, α is the tire slip angle, and κ_c and α_c are the critical slip factors for the longitudinal and lateral models, which are parameters employed by the tire model. For more details of this tire model, the reader is referred to [2] or [3].

The parameters for the normal and frictional tire force models are the same for the four wheels and are gathered in Table 6.

Tire radius	**Normal force parameters**		**Frictional force parameters**			
(m)	k_m ($\mathrm{N\,m^{-1}}$)	c_m ($\mathrm{N\,s\,m^{-1}}$)	κ_c	α_c	μ_x	μ_y
0.30253	60430	100	0.8	0.2	0.7	0.7

Table 6: Buggy model: tire force parameters.

1.3 Mass properties

All bodies in the multibody system are regarded as rigid bodies, with the mass, center of mass and the inertia tensor displayed in Table 7 (the two later expressed in the local reference frame of each body). All magnitudes are given in the SI units system.

Body	Mass (kg)	Local CoM (m)			Inertia tensor (kg m^2)
Chassis	108.384	-1.27085	-0.00031	-0.24711	$\begin{bmatrix} 21.48575 & -0.07787 & -2.13974 \\ -0.07787 & 68.88748 & -0.02568 \\ -2.13974 & -0.02568 & 62.74836 \end{bmatrix}$
Inferior frontal wishbone (x2)	4.194	-0.10895	0.18193	0.00000	$\begin{bmatrix} 0.02019 & 0.01689 & 0.00000 \\ 0.01689 & 0.06139 & 0.00000 \\ 0.00000 & 0.00000 & 0.08092 \end{bmatrix}$
Superior frontal wishbone (x2)	4.194	-0.10895	0.18193	0.00000	$\begin{bmatrix} 0.02019 & 0.01689 & 0.00000 \\ 0.01689 & 0.06139 & 0.00000 \\ 0.00000 & 0.00000 & 0.08092 \end{bmatrix}$
Frontal right upright	3.306	-0.01772	0.06272	0.02456	$\begin{bmatrix} 0.01153 & 0.00179 & 0.00248 \\ 0.00179 & 0.01466 & -0.00180 \\ 0.00248 & -0.00180 & 0.00754 \end{bmatrix}$
Frontal left upright	3.306	-0.01772	-0.06272	0.02456	$\begin{bmatrix} 0.01153 & -0.00179 & 0.00248 \\ -0.00179 & 0.01466 & 0.00180 \\ 0.00248 & 0.00180 & 0.00754 \end{bmatrix}$
Steering rack	0.572	0.00000	0.00000	0.00000	$\begin{bmatrix} 0.00001 & 0.00000 & 0.00000 \\ 0.00000 & 0.01977 & 0.00000 \\ 0.00000 & 0.00000 & 0.01977 \end{bmatrix}$
Steering rod (x2)	0.291	0.00000	0.00000	0.00000	$\begin{bmatrix} 0.00001 & 0.00000 & 0.00000 \\ 0.00000 & 0.00248 & 0.00000 \\ 0.00000 & 0.00000 & 0.00248 \end{bmatrix}$
Rear wishbone (x2)	2.653	0.06000	0.00000	0.00000	$\begin{bmatrix} 0.05489 & 0.00000 & 0.00000 \\ 0.00000 & 0.06371 & 0.00000 \\ 0.00000 & 0.00000 & 0.00923 \end{bmatrix}$
Rear upright (x2)	5.994	-0.07488	0.02003	0.00002	$\begin{bmatrix} 0.04119 & 0.00896 & 0.00001 \\ 0.00896 & 0.01488 & -0.00001 \\ 0.00001 & -0.00001 & 0.04489 \end{bmatrix}$
Front wheel (x2)	15.140	0.06214	0.00000	0.00000	$\begin{bmatrix} 0.52811 & 0.00000 & 0.00000 \\ 0.00000 & 0.29466 & 0.00000 \\ 0.00000 & 0.00000 & 0.29466 \end{bmatrix}$
Rear wheel (x2)	16.087	0.06513	0.00000	0.00000	$\begin{bmatrix} 0.53419 & 0.00000 & 0.00000 \\ 0.00000 & 0.29957 & 0.00000 \\ 0.00000 & 0.00000 & 0.29957 \end{bmatrix}$

Table 7: Buggy model: mass properties of bodies.

2 Step descent maneuver: dynamic description and solution

The maneuver consists in a 4.5 second dynamic simulation of the buggy vehicle with its steering system blocked in order to keep the four wheels aligned in the forward direction. In this regard, the coordinate s_{50} associated to the steering system is guided with:

$$s_{50} = f(t) = 0.326 \tag{4}$$

$$\dot{s}_{50} = \dot{f}(t) = 0 \tag{5}$$

$$\ddot{s}_{50} = \ddot{f}(t) = 0 \tag{6}$$

At 5.5 m from the origin, the vehicle encounters an abrupt step descent of 1 cm, which means that the profile of the z coordinate of the plane defining the ground changes from 0 to −1 cm, according to Figure 2:

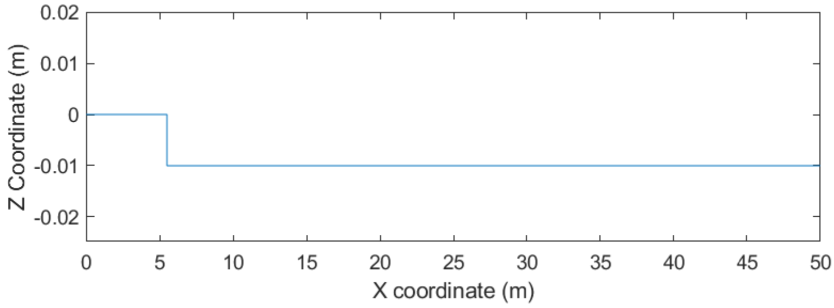

Figure 2: Evolution of Z coordinate of the surface of the ground.

The initial configuration of the system is given by the position and velocity of the 14 degrees of freedom listed in Table 8.

N	Variable	Initial position	Initial velocity
1	p_1	0 m	3 m s^{-1}
2	p_{1y}	0 m	0 m s^{-1}
3	p_{1z}	0.320 00 m	0 m s^{-1}
4	v_{1y}	1	0 s^{-1}
5	v_{1z}	0	0 s^{-1}
6	v_{2z}	0	0 s^{-1}
7	s_{10}	0.674 56 m	0 m s^{-1}
8	s_{20}	0.674 56 m	0 m s^{-1}
9	s_{30}	0.708 22 m	0 m s^{-1}
10	s_{40}	0.708 22 m	0 m s^{-1}
11	a_{10}	0 rad	11 rad s^{-1}
12	a_{20}	0 rad	11 rad s^{-1}
13	a_{30}	0 rad	11 rad s^{-1}
14	a_{40}	0 rad	11 rad s^{-1}

Table 8: Buggy model: initial configuration of the degrees of freedom in the step-descent maneuver.

The evolution of position, velocity and acceleration of point 1 over time is displayed in Figure 3, and the same results can be found on the data files *dynamics_std_pt1_pos.csv*, *dynamics_std_pt1_vel.csv* and *dynamics_std_pt1_acel.csv*.

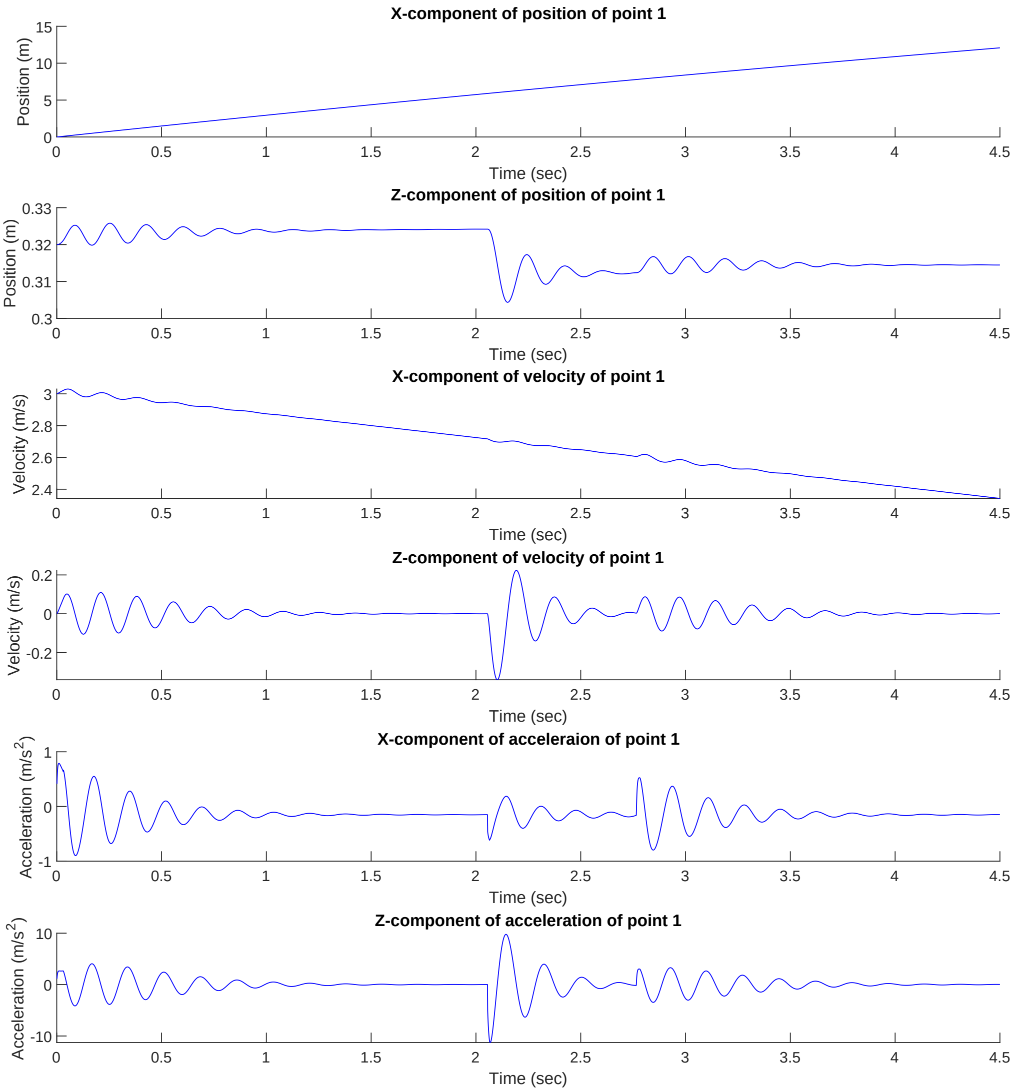

Figure 3: Evolution of position, velocity and acceleration of point 1 over time in the step descent maneuver.

3 Step descent maneuver: sensitivity problem description and solution

For the sensitivity analysis of this maneuver, the following objective function is considered:

$$\psi = \int_{t_0}^{t_F} \ddot{r}_{1_z}^2 \, \mathrm{d}t \tag{7}$$

wherein $\ddot{r}_{1_z}$ denotes the Z-component of the acceleration of point 1, located in the chassis. The evaluation of the accelerations that the driver would experiment during the descent of the step included in the objective function gives a measure of comfort and is related to vibration measurements according to norm ISO 2631-1 [4]

The parameters selected for the sensitivity analysis in both maneuvers are:

$$\boldsymbol{\rho} = \begin{bmatrix} k_f & c_f & k_r & c_r & m_c \end{bmatrix}^{\mathrm{T}} \tag{8}$$

where k_f and c_f are the stiffness and damping coefficients of the front suspensions, k_r and c_r denote the stiffness and damping coefficients of the rear suspensions and m_c represents the mass of the chassis. At the initial instant of time, none of the parameters affect the degrees of freedom of the system (see Table 8), thus their initial sensitivities at position and velocity level are all null.

The reference results obtained for the gradient of the objective function (7) are listed in Table 9.

$\boldsymbol{\psi}^{Ref}$	$(\boldsymbol{\psi})_{k_f}^{\prime Ref}$	$(\boldsymbol{\psi})_{c_f}^{\prime Ref}$	$(\boldsymbol{\psi})_{k_r}^{\prime Ref}$	$(\boldsymbol{\psi})_{c_r}^{\prime Ref}$	$(\boldsymbol{\psi})_{m_c}^{\prime Ref}$
16.4163	2.06×10^{-4}	9.34×10^{-4}	-3.90×10^{-5}	7.53×10^{-4}	4.06×10^{-2}

Table 9: Objective function gradient for the step descent maneuver

A solution to this problem is considered to be converged if the error evaluated through equations (9) and (10) is below a tolerance of 1.5×10^{-1}:

$$error = \sum_{i=1}^{5} k_i \epsilon_i \tag{9}$$

with $k_i = 10^2$ for $i = 5$ and $k_i = 10^4$ for $i \neq 5$, and

$$\epsilon_1 = |(\boldsymbol{\psi})'_{k_f} - (\boldsymbol{\psi})_{k_f}^{\prime Ref}| \tag{10a}$$

$$\epsilon_2 = |(\boldsymbol{\psi})'_{c_f} - (\boldsymbol{\psi})_{c_f}^{\prime Ref}| \tag{10b}$$

$$\epsilon_3 = |(\boldsymbol{\psi})'_{k_r} - (\boldsymbol{\psi})_{k_r}^{\prime Ref}| \tag{10c}$$

$$\epsilon_4 = |(\boldsymbol{\psi})'_{c_r} - (\boldsymbol{\psi})_{c_r}^{\prime Ref}| \tag{10d}$$

$$\epsilon_5 = |(\boldsymbol{\psi})'_{m_c} - (\boldsymbol{\psi})_{m_c}^{\prime Ref}| \tag{10e}$$

(10f)

The computational times provided for each proposed solution should be the sum of CPU times of both dynamics and sensitivity.

The file *results.txt*, attached to the solution, provides the sensitivities the objective function in the same order of Table 9.

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